Properties of fractional linear transformations

(1) Conformal as maps from $\mathbb{C}\cup\{\infty\} \to \mathbb{C}\cup\{\infty\}$, hence angles preserving.

(2) f, g are fractional linear transformations
$\Rightarrow$ $f\circ g$ is a fractional linear transformation.

(3) fractional linear transformation is a composition of translations, dilations and inversions.

(4) fractional linear transformations map "straight lines & circles" to "straight lines or circles".

Pf: (1) Clearly $f(z)=\dfrac{az+b}{cz+d}$ has derivatives

$$f'(z)=\frac{ad-bc}{(cz+d)^2}\neq 0 \quad \text{for } z\neq -\frac{d}{c}.$$

(we omit the discussion at $z=-\frac{d}{c}$ and $z=\infty$)

Also, clearly $g(w)=\dfrac{dw-b}{-cw+a}$ is the inverse of f

(Note: $z=-\frac{d}{c} \leftrightarrow w=\infty$, $z=\infty \leftrightarrow w=\frac{a}{c}$)

$\therefore$ f is conformal (from $\mathbb{C}\cup\{\infty\} \to \mathbb{C}\cup\{\infty\}$)

(2) If $f(z)=\dfrac{az+b}{cz+d}$, $ad-bc\neq 0$

$g(z)=\dfrac{kz+l}{mz+n}$, $kn-lm\neq 0$

Then $$f\circ g(z) = \frac{a\left(\frac{kz+l}{mz+n}\right)+b}{c\left(\frac{kz+l}{mz+n}\right)+d} = \frac{(ak+bm)z+(al+bn)}{(ck+dm)z+(cl+dn)}$$

Note that $\begin{pmatrix} ak+bm & al+bn \\ ck+dm & cl+dn \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} k & l \\ m & n \end{pmatrix}$

$$\begin{aligned} \therefore\ & (ak+bm)(cl+dn) - (al+bm)(ck+dm) \\ &= \det\begin{pmatrix} ak+bm & al+bn \\ ck+dm & cl+dn \end{pmatrix} \\ &= \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} \det\begin{pmatrix} k & l \\ m & n \end{pmatrix} \\ &= (ad-bc)(kn-lm) \neq 0 . \end{aligned}$$

$\therefore$ $f\circ g$ is a fractional linear transfomation

(3) $f(z) = \dfrac{az+b}{cz+d}$, $ad-bc \neq 0$

If $c=0$, then $d \neq 0$ & $f(z) = \left(\frac{a}{d}\right)z + \left(\frac{b}{d}\right)$

ie. $z \underset{\substack{\uparrow \\ \text{dilation} \\ (a\neq 0)}}{\longmapsto} \left(\frac{a}{d}\right)z \underset{\substack{\uparrow \\ \text{translation}}}{\longmapsto} \left[\left(\frac{a}{d}\right)z\right] + \left(\frac{b}{d}\right) = f(z)$

If $c \neq 0$, then

$$f(z) = \frac{az+b}{cz+d} = \frac{1}{c} \cdot \frac{az+b}{z+\frac{d}{c}}$$

$$= \frac{1}{c}\left[\frac{a(z+\frac{d}{c}) - \frac{ad}{c} + b}{z+\frac{d}{c}}\right]$$

$$= \frac{1}{c}\left[a - \frac{\frac{ad}{c} - b}{z+\frac{d}{c}}\right]$$

$$= \frac{a}{c} - \frac{(ad-bc)}{c^2}\cdot\frac{1}{z+\frac{d}{c}}$$

i.e. $\underset{\text{translation}}{z \mapsto z+\frac{d}{c}} \underset{\text{inversion}}{\to} \frac{1}{z+\frac{d}{c}} \underset{\text{dilation}}{\mapsto} -\frac{(ad-bc)}{c^2}\frac{1}{z+\frac{d}{c}}$

$\underset{\text{translation.}}{\mapsto} \frac{a}{c} - \frac{ad-bc}{c^2}\frac{1}{z+\frac{d}{c}}$

(4) Note that translations and dilations map straight lines to straight lines, and circles to circles. Then because of (3), we only need to prove (4) for inversion $z \mapsto \frac{1}{z}$.

let $z = x+iy$ & $w = s+it = \frac{1}{z}$

then $s+it = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$

i.e. $\begin{cases} s = \frac{x}{x^2+y^2} \\ t = -\frac{y}{x^2+y^2} \end{cases}$ (Inversion as a mapping from $\mathbb{R}^2\setminus\{0\} \to \mathbb{R}^2\setminus\{0\}$)

Also $wz=1 \Rightarrow |w|^2|z|^2=1$, ie. $\left(\Rightarrow \begin{cases} x = \dfrac{s}{s^2+t^2} \\ y = \dfrac{-t}{s^2+t^2} \end{cases} \right)$

$$s^2+t^2 = \frac{1}{x^2+y^2}$$

Now let $L: ax+by+c=0$ be a straight line

Then $\dfrac{as}{s^2+t^2} - \dfrac{bt}{s^2+t^2} + c = 0$

ie. $c(s^2+t^2)+as-bt=0$

If $c=0$ (ie. L passing thro the origin),

the image of L is the straight line

$L': as-bt=0$. (in (s,t)-plane)

If $c\neq 0$ (ie. L not passing thro the origin)

$\therefore$ the image of L is the circle

$C': s^2+t^2+\left(\frac{a}{c}\right)s - \frac{b}{c}t = 0$ (in (s,t)-plane)

Now let $C: x^2+y^2+ax+by+c=0$ be a circle.

Then we have $\dfrac{1}{s^2+t^2} + \dfrac{as}{s^2+t^2} - \dfrac{bt}{s^2+t^2} + c = 0$

$\Rightarrow c(s^2+t^2)+as-bt+1=0$.

If $c=0$, the image of C is a straight line

$L': as-bt+1=0$

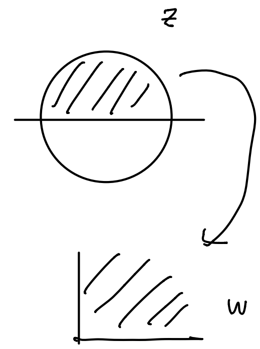
If $c\neq 0$, the image of C is a circle

$C': s^2+t^2+\left(\frac{a}{c}\right)s-\left(\frac{b}{c}\right)t+\frac{1}{c}=0$. ※

Eg3 (of the Text book)

$f(z) = \dfrac{1+z}{1-z} : \{ z = x+iy : |z|<1 \text{ and } y>0 \} = \mathbb{D}^+$

$\longrightarrow \{ w = u+iv : u>0 \text{ and } v>0 \} = S$



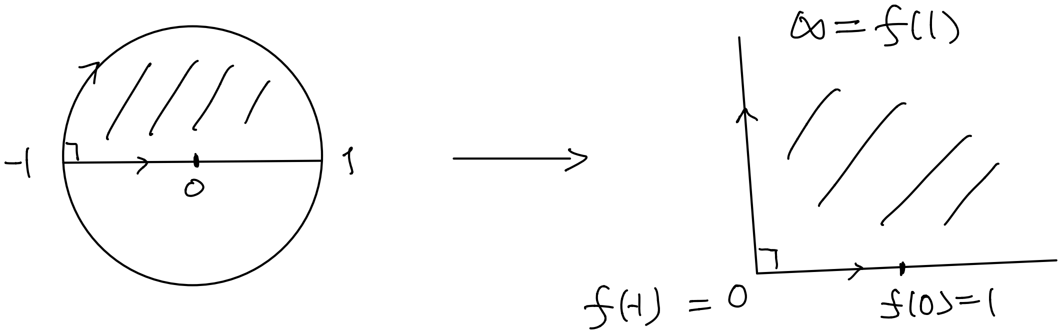
is conformal.

Note: f is a fractional linear transformation

$f(z) = \dfrac{z+1}{-z+1}$ with $1\cdot 1 - (1)(-1) = 2 \neq 0$

$\therefore$ f is injective, hence remain to show $f(\mathbb{D}^+) = S$.

Observe that $f(-1) = 0$, $f(0) = 1$, $f(1) = \infty$



By property (4) of fractional linear transformation,
the real line segment between -1 & 1
maps to part of a straight line or a circle.
Since it passes through $f(-1) = 0$, $f(0) = 1$
& $f(1) = \infty$, it is the positive real axis.

Similarly, the upper semi-circle maps to part of a straight line or a circle passing through 0 and ∞, and hence must be a straight line.

Since the angles from $[-1,1]$ to the semi-circle is $\frac{\pi}{2}$, the angle from the positive x-axis to the image straight line of the semi-circle is also $\frac{\pi}{2}$ (f conformal)

$\therefore$ the image of the upper semi-circle is the positive y-axis.

(Positivity can also be confirmed by $f(i) = \frac{1+i}{1-i} = i$)

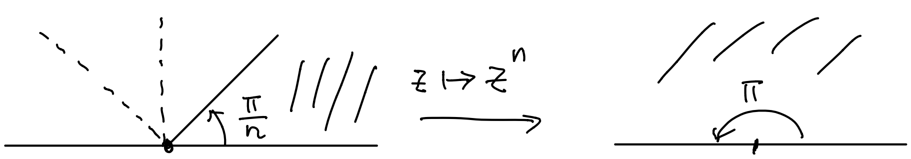
This shows that $f(D^+) = S$ (as f is conformal : $\mathbb{C}\cup\{\infty\} \to \mathbb{C}\cup\{\infty\}$)

(Of course, all these can be proved by using coordinates as in the Textbook)

<u>Eg2</u> (Of the Textbook)

For $n = 1, 2, 3, \cdots$, $z \mapsto z^n : S \to \mathbb{H}$ is conformal,

where $S = \{ z \in \mathbb{C} : 0 < \arg(z) < \frac{\pi}{n} \}$

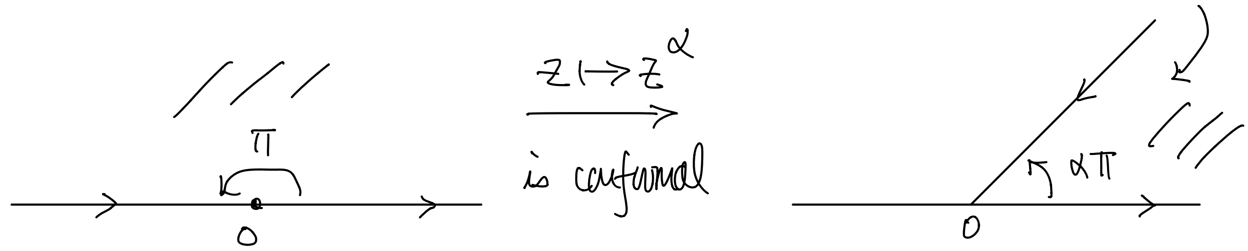


Inverse $w \mapsto w^{\frac{1}{n}} : \mathbb{H} \to S$

where $w^{\frac{1}{n}} = e^{\frac{1}{n}\log w}$ with $\log w$ = principal branch

More generally, for $0 < \alpha < 2$ $\quad (0 < \frac{1}{n} \leq 2)$

$z \in \mathbb{H}$ $\qquad\qquad$ $S = \{ w \in \mathbb{C} : 0 < \arg(w) < \alpha\pi \}$



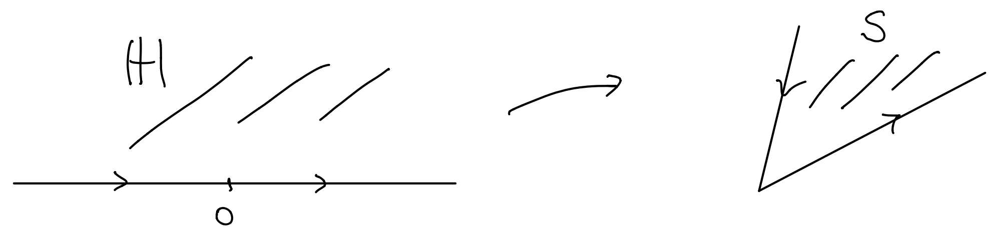
with inverse $w \mapsto w^{\frac{1}{\alpha}} = e^{\frac{1}{\alpha}\log w}$

where branch of $\log w$ s.t. $0 < \arg w < \alpha\pi$.

(Boundary behavior as in the figure.)

<u>Conclusion</u>: One can map $\mathbb{H}$ conformally to any (infinite) sector in $\mathbb{C}$

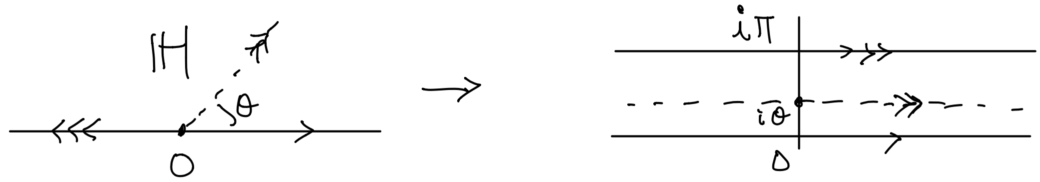
(by composing the maps here with translations & rotations.)



<u>Eg 4</u>: $z \mapsto \log z$ branch defined by deleting $\{x < 0\}$

(i.e. $-\pi < \arg z < \pi$)

maps $\mathbb{H}$ conformally to strip $\{ w = u + iv : 0 < v < \pi \}$.



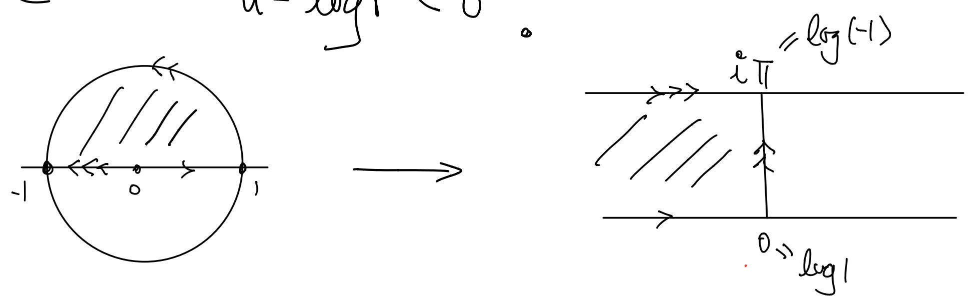
By the choice of the branch, for $z = re^{i\theta}$, $0 < \theta < \pi$

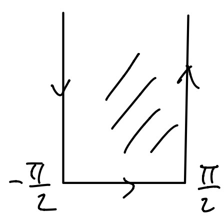
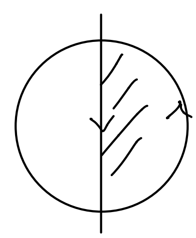
$$\log z = \log r + i\theta$$

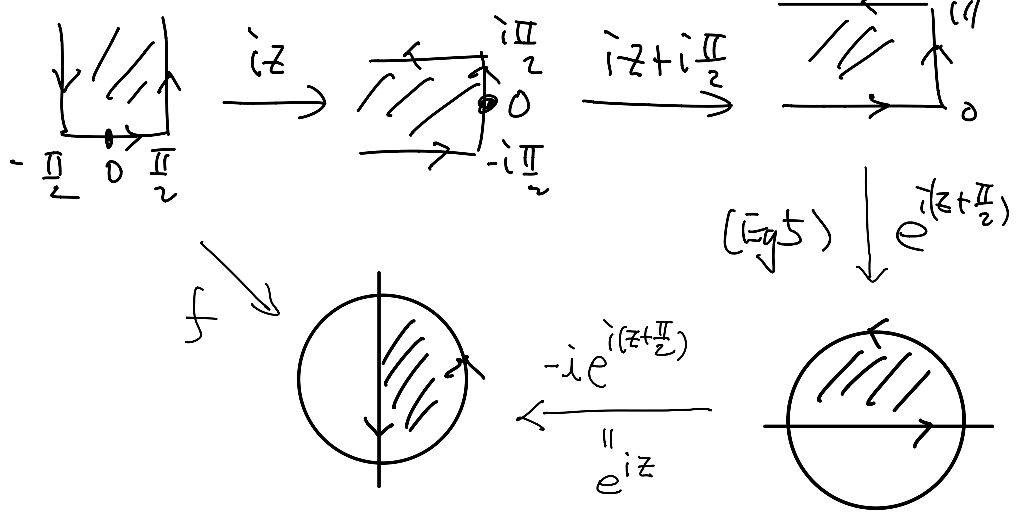
$\therefore \quad u = \log r \in \mathbb{R}$ and $v = \theta \in (0, \pi)$

The inverse is $w \mapsto e^w$.

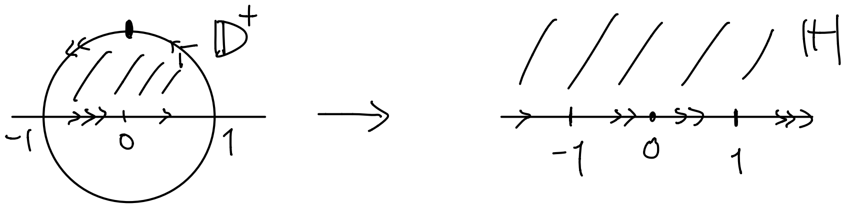
Eg5 Same $z \mapsto \log z$ maps $\mathbb{D}^+ = \{z = x+iy : |z| < 1, y > 0\}$ conformally to <u>half strip</u> $\{w = u + iv : u < 0, 0 < v < \pi\}$, since $u = \log r < 0$.



Eg6: $f(z) = e^{iz}$ maps  conformally onto 

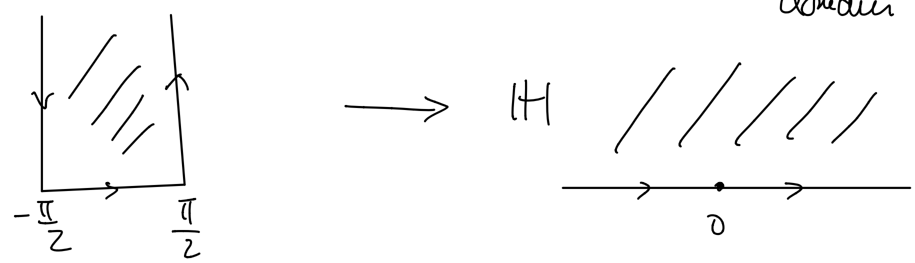


<u>Eg 7</u> (Ex5) $f(z) = -\frac{1}{2}\left(z+\frac{1}{z}\right)$ maps conformally



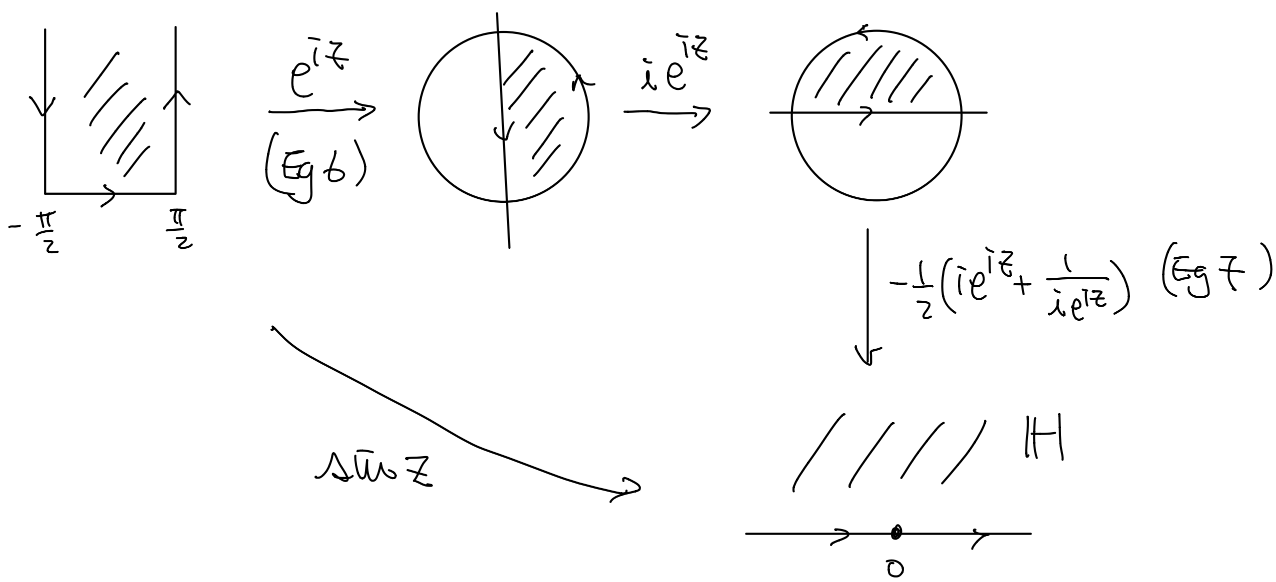
(misprint in the textbook concerning the boundary behaviour, confused with $z \mapsto \frac{1}{2}\left(z+\frac{1}{z}\right)$)

<u>Eg 8</u> $f(z) = \sin z$ maps conformally (misprint in Textbook, confused domain and target)



Note $f(z) = \sin z = \dfrac{e^{iz}-e^{-iz}}{2i} = -\frac{1}{2}\left(\dfrac{-e^{iz}}{i} + \dfrac{e^{-iz}}{i}\right)$

$$= -\frac{1}{2}\left(ie^{iz} + \frac{1}{ie^{iz}}\right)$$



1.2 The Dirichlet Problem in a Strip

Dirichlet Problem in the open set Ω consists of solving

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = f & \text{on } \partial\Omega \end{cases}$$

where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the Laplacian (operator)

f = given (continuous) function on $\partial\Omega$.

(i.e. Dirichlet problem = Boundary Value Problem for the Laplace equation)

Known Fact: Solution to Dirichlet Problem in the unit disk $\mathbb{D}$.

Recall: using polar coordinates

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

Let f be a continuous function on $\partial\mathbb{D} = S^1$.

Then f can be represented as a (periodic) function of θ

$$f(\theta), \quad 0 \le \theta \le 2\pi.$$

Then the unique solution to $\begin{cases} \Delta u = 0 & \text{in } \mathbb{D} \\ u = f & \text{on } \partial\mathbb{D} = S^1 \end{cases}$

is given by $$u(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - \varphi) f(\varphi)\, d\varphi$$

where $$P_r(\theta) = \frac{1 - r^2}{1 - 2r\cos\theta + r^2}$$

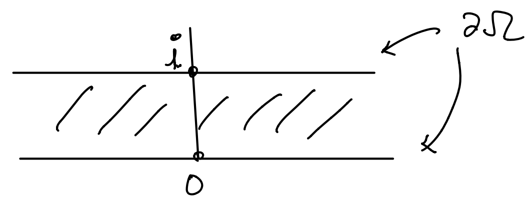
(See Textbook for reference)

In this section, we illustrate how to use conformal maps and the solution of Dirichlet problem in the unit disc to solve Dirichlet Problem in a more general domain Ω in $\mathbb{C}$.

Lemma 1.3 $F: U \to V$ holo. (U, V open in $\mathbb{C}$)
If $u: V \to \mathbb{C}$ is harmonic (ie. $\Delta u = 0$), then $u \circ F: U \to \mathbb{C}$ is harmonic.

Pf Easy exercise using Chain rule and Cauchy-Riemann equation. (Or observing that $\exists$ holo. G on U s.t. $\mathrm{Re}\, G = u$.)

Dirichlet Problem in the strip $\Omega = \{x+iy : x \in \mathbb{R}, 0 < y < 1\}$



Then boundary $\partial\Omega$ of Ω consists of two components
$L_0 = \{x+iy : y = 0\}$ & $L_1 = \{x+iy : y = 1\}$

Let $f_0 : L_0 \to \mathbb{R}$ and $f_1 : L_1 \to \mathbb{R}$ be continuous functions (and represented as functions of x only)

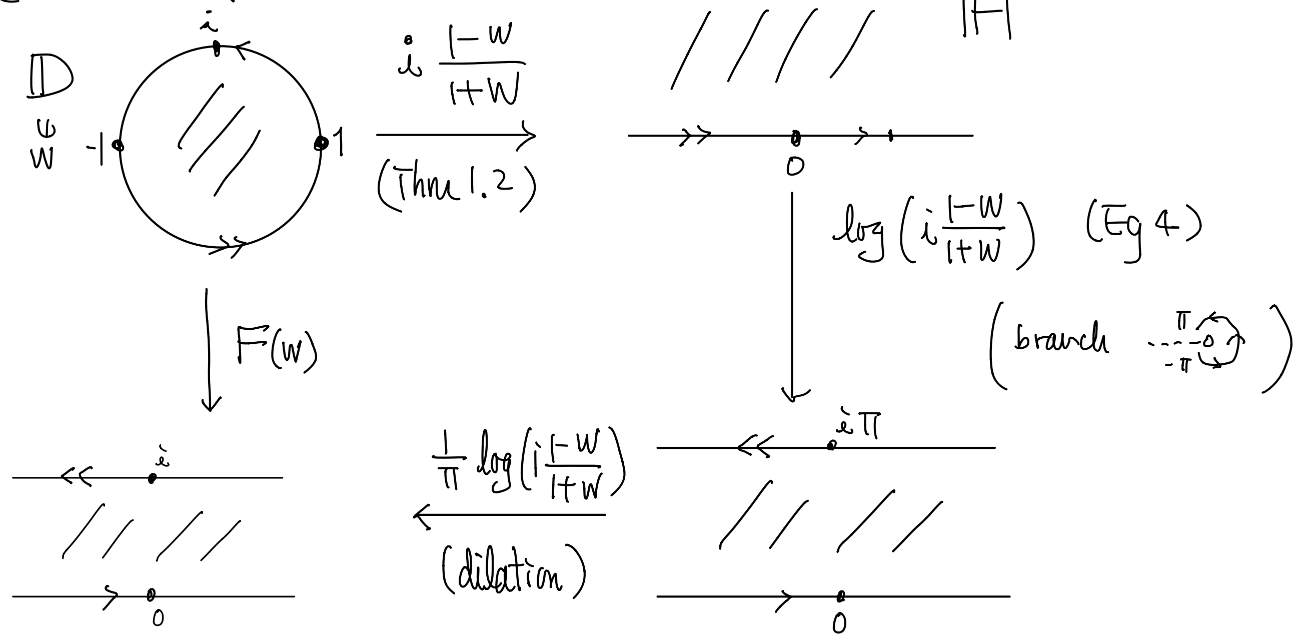
We need to find $u(x,y)$ such that $\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u(x,0) = f_0(x) \\ u(x,1) = f_1(x) \end{cases}$

For techincal reason, let consider special cases that

$$\lim_{|x|\to\infty} f_0(x) = \lim_{|x|\to\infty} f_1(x) = 0 .$$

<u>Step 1</u>: Find a conformal map $F: \mathbb{D} \to \Omega$.

Recall that



$\therefore$ $\quad F(w) = \frac{1}{\pi}\log\left(i\frac{1-w}{1+w}\right)$ maps $\mathbb{D}$ conformally onto Ω.

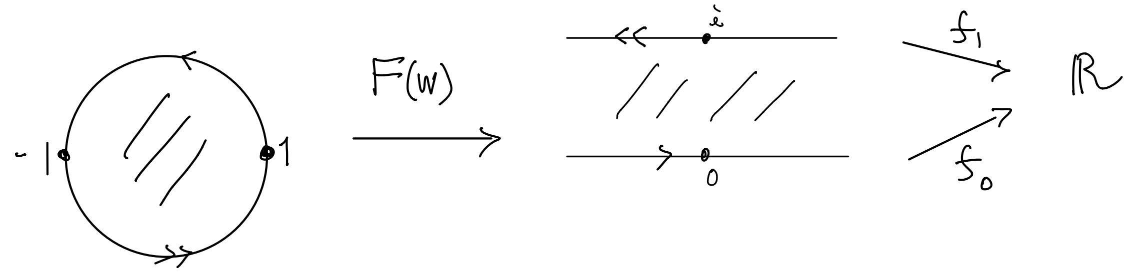
Easy calculation $\Rightarrow$

$$G(z) = \frac{i - e^{\pi z}}{i + e^{\pi z}} : \Omega \to \mathbb{D}$$

is the inverse $(G = F^{-1})$

Boundary behaviour: $\varphi: -\pi \to 0 \iff F(e^{i\varphi}): i+\infty \to i-\infty$

$\varphi: 0 \to \pi \iff F(e^{i\varphi}): -\infty \to +\infty$ (x-axis)



Define $\tilde{f} : S^1 = \partial\mathbb{D} \to \mathbb{R}$ by

$$\tilde{f}(\varphi) = \begin{cases} f_0(F(e^{i\varphi})) , & 0 < \varphi < \pi \\ f_1(F(e^{i\varphi}) - i) , & -\pi < \varphi < 0 \\ 0 , & \varphi = 0, \pm\pi \end{cases}$$

Then by $\lim_{|x|\to\infty} f_0(x) = \lim_{|x|\to\infty} f_1(x) = 0$, $\tilde{f}$ is continuous.

Using the solution to the Dirichlet problem in the unit disc $\mathbb{D}$,

$$\tilde{u}(w) = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(\theta - \varphi) \tilde{f}(\varphi) d\varphi$$

is a harmonic function in $\mathbb{D}$ with boundary value

$$\tilde{u}\big|_{\partial\mathbb{D}} = \tilde{f} .$$

Then Lemma 1.3 $\Rightarrow$ $\quad u = \tilde{u} \circ G : \Omega \to \mathbb{R} \ (\subset \mathbb{C})$

is the solution to the Dirichlet problem in the strip Ω.

More explicitly, we have

$$\boxed{u(x,y) = \frac{\sin \pi y}{2} \left(\int_{-\infty}^{\infty} \frac{f_0(x-t)}{\cosh(\pi t) - \cos \pi y} dy + \int_{-\infty}^{\infty} \frac{f_1(x-t)}{\cosh(\pi t) + \cos \pi y} dy \right) \quad (0 < y < 1)}$$

(Details omitted see Ex 7 & discussion on page 216 in the Textbook)